Walls, Chambers, Surfaces

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We want to investigate the wall and chamber structure for the moduli space of stability conditions on a surface, which surprisingly can be done with rather elementary techniques. We are following Antony Maciocas paper, which was referenced for the proof of the wall and chamber structure.

1 Plane

For $\omega \in \mathrm{NS}(X)$ and $B \in \mathrm{NS}_{\mathbb{R}}(X)$ we recall, that we can define abelian subcategories $\mathcal{A}_{\beta,\omega}$ by tilting

$$T_{B,\omega} = \{\mathcal{E} \in \mathrm{Coh}(X) | \mathcal{E} \text{ is torsion } \mu_\omega > B \cdot \omega\} \tag{1}$$

$$F_{B,\omega} = \{\mathcal{E} \in \mathrm{Coh}(X) | \mathcal{E} \text{ is torsion-free } \mu_\omega \leq B \cdot \omega\} \tag{2}$$

We make these into stability conditions via

$$Z_{\omega,B} = -\exp(-B - i\omega) \cdot v \tag{3}$$

Remark 1.1. This agrees with our previous notion of tilting. We recall a B-twisted chern character is

$$\mathrm{ch}^B(\mathcal{E}) = \mathrm{ch}(\mathcal{E})e^{-B} = \mathrm{ch}(\mathcal{E})(1 - B + \frac{1}{2}B^2 - ...).$$

Then the twisted first chern class $c_1^B(\mathcal{E})$ is given by the degree 1 part (in cohomology), i.e.

$$c_1^B(\mathcal{E}) = c_1(\mathcal{E}) - \mathrm{rk}(\mathcal{E})B.$$

We recall the notion of B-twisted stability

$$\mu_{\omega,B} = \frac{c_1^B(\mathcal{E})\omega}{\mathrm{rk}\,\mathcal{E}} \tag{4}$$

In Macris notes the notion of tilting is given by the condition

$$\mu_{\omega,B} > 0 \Longleftrightarrow \mu_\omega - B \cdot \omega > 0 \Longleftrightarrow \mu_\omega > B \cdot \omega, \tag{5}$$

where we are supressing the fact that we are actually asking this equation to hold for any semistable factor.

Remark 1.2. For any $t > 0$, $\mathcal{A}_{t\omega,B} = \mathcal{A}_{\omega,B}$. So this subcategory is actually an entire ray in the space of stability conditions.

Let $K_{num}(X)$ be the numerical Grothendieck group and we fix a class $v \in K_{num}(X)$ and we assume $v = v(a)$ for some $a \in D(X)$. For a coherent sheaf A on X we let $v(A)$ denote its image in A. This can be extended to the derived category by defining $v(a) = \sum_i (-1)^i v(A^i)$ where $a \in D(X)$ and we use the convenient abbreviated notation $A^i = H^i(a)$. We can also make the identification

$$\begin{aligned} \mathrm{ch} : K_{num}(X) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}[\frac{1}{2}] \\ v(a) \mapsto (r(a), c_1(a), \mathrm{ch}_2(a)[X]) \end{aligned} \tag{6}$$

First of all to get some handle on the situation we should be restricting us to a subspace on this space. We construct it as follows. Consider $(A_{B,\omega}, Z_{B,\omega}) \in \mathrm{Stab}(X)$. We can now also consider the ray

$$R_{B,\omega} = \{(\mathcal{A}_{\omega,B}, Z_{t\omega,B}) | 0 < t \in \mathbb{R}\} \tag{7}$$

Furthermore, if $B \neq 0$ we can construct the plane

$$P_{B,\omega} = \{(\mathcal{A}_{sB,\omega}, Z_{sB,t\omega}) | 0 < t \in \mathbb{R}, s \in \mathbb{R}\} \tag{8}$$

We can simplify this by making a choices, which then lets us identify this with the (α, β)-plane found in Macris notes. We write $B = b\omega + \gamma$ and we get

$$\Omega_{B,\omega} = \{(\mathcal{A}_{\omega,s\omega+u\gamma}, Z_{\omega,s\omega+u\gamma}) | s, t, u \in \mathbb{R}, t > 0\}. \tag{9}$$

If we want to identify this with Macris notation, we set $B_0 \coloneqq u\gamma \in \mathrm{NS}_{\mathbb{Q}}(X)$, $\alpha \coloneqq t$ and $H \coloneqq \omega \in \mathrm{NS}(X)$.

Definition 1.1. *Let H, B_0 as above. The (α, β)-plane is the set of stability conditions for $\beta + i\alpha \in \mathbb{H}$, where $\sigma_{\alpha H, B_0 + \beta H}$.*

Remark 1.3. There is also a more general definition of a plane, but if we choose the picard number to be 1. This is the most general plane one could look at.

Remark 1.4. By abuse, we will drop the H, B_0 from the notation and simply refer to it as $\sigma_{\alpha,\beta}$.

2 Walls

Next we can move on towards the definition of a wall. L.

Definition 2.1. *Let $w \in K_{num}(X) \backslash \langle v \rangle$. We say w is critical for v if each of the following conditions hold:*

(a) There exists $\sigma \in \mathrm{Stab}(X)$ and objects $a, b \in \mathcal{A}_\sigma$ with an injection $b \to a$

(b) $v(a) = a$ and $v(b) = w$

(c) $\mu_Z(a) = \mu_Z(b)$

We call the critical subset $W_w^v \subset \mathrm{Stab}(X)$ *a wall. We can drop the first two conditions and we call the "critical" set* $PW_w^v \subset \mathrm{Stab}(X)$ *a pseudo-wall.*

Remark 2.1. It is easier to study pseudo-walls and the walls will be a subset of them, so most theorems will also hold for the walls.

We want to prove 6.22 in Macris notes.

3 Wall Structure

We start by first proving, that all Walls are semicircles. We fix a class $v \in K_{num}(X)$ such that it has positive discriminant, i.e.

$$c_1(v)^2 \geq 2r(v)\,\mathrm{ch}_2(v) \tag{10}$$

Proposition 3.1. *The walls for stability conditions are given by semicircles centered on the* t*-axis.*

Proof. We fix $\omega \in \mathrm{NS}(X)$, $\gamma \in \mathrm{NS}_{\mathbb{Q}}(X)$ such that $\langle \gamma, \omega \rangle = 0$, and $v = (x, \theta, z)$. We write

$$\mathrm{NS}_{\mathbb{Q}}(X) = \langle \omega \rangle \oplus \langle \gamma \rangle \oplus \langle \gamma, \omega \rangle^{\perp}. \tag{11}$$

Decompose $\theta = y_1\omega + y_2\gamma + \alpha$, where $\alpha \in \langle B_0, H \rangle^{\perp}$. Consider the slope stability function

$$\begin{aligned} \mu_Z(v) &= -\frac{\mathrm{Re}\, Z_{\omega,B}(v)}{\mathrm{Im}\, Z_{\omega,B}(v)} \\ &= \frac{z - \theta \cdot \beta + \frac{x}{2}(B^2 - \omega^2)}{(\theta - r \cdot B)\omega} \end{aligned} \tag{12}$$

Then for $Z = Z_{t\omega, s\omega + u\gamma}$

$$\mu_Z(v) = \frac{z - sy_1\omega^2 - uy_2\gamma^2 + \frac{x}{2}(s^2\omega^2 + u^2\gamma^2 - t^2\omega^2)}{(y_1\omega - y_2\gamma + \alpha - x(s\omega - u\gamma))t\omega} \tag{13}$$

alpha and γ are orthogonal to ω, so we get some cancellation in the denominator

$$\mu_Z(v) = \frac{z - sy_1\omega^2 - uy_2\gamma^2 + \frac{x}{2}(s^2\omega^2 + u^2\gamma^2 - t^2\omega^2)}{(y_1 - xs)t\omega^2} \tag{14}$$

In the same manner we compute for $w = (r, c_1\omega + c_2\gamma + \alpha', \chi)$

$$\mu_Z(w) = \frac{\chi - (c_1 s\omega^2 + c_2 u\gamma^2) + \frac{r}{2}(s^2\omega^2 + u^2\gamma^2 - t^2\omega^2)}{(c_1 - rs)t\omega^2} \tag{15}$$

Next we compute the difference $\mu_Z(w) - \mu_Z(v)$

$$\frac{-(c_1 - rs)(z - sy_1\omega^2 - uy_2\gamma^2) + \frac{x}{2}(s^2\omega^2 - u\gamma^2 - t^2\omega^2)}{(y_1 - xs)(c_1 - rs)t\omega^2} \tag{16}$$

$$+\frac{+(y_1 - xs)(\chi - sc_1\omega^2 - uc_2\gamma^2) + \frac{r}{2}(s^2\omega^2 + u^2\gamma^2 - t^2\omega^2)}{(y_1 - xs)(c_1 - rs)t\omega^2} \tag{17}$$

if you multiply it by 2 collect the s^2 terms the t^2 terms the $2s$ terms and the constant terms you get

$$s^2\omega^2(xc_1 - y_1r) + t^2\omega^2(xc_1 - y_1r) - 2s(\chi x - uc_2\gamma^2 x + y_2\gamma^2 x) - D' \tag{18}$$

To make it look like a circle we can now divide by $\omega^2(xc_1 - y_1r)$

$$\begin{aligned} 0 = 2\frac{\mu_Z(w) - \mu_Z(v)}{\omega^2(xc_1 - y_1r)} &= (s^2 + t^2 - 2sC - D) && (19)\\ &= (s^2 + t^2 - 2sC + C^2 - C^2 - D) && (20)\\ &= ((s - C)^2 + t^2 - C^2 - D), && (21)\\ & && (22) \end{aligned}$$

where

$$C = \frac{x\chi - rz - u\gamma^2(xc_2 - ry_2)}{g(xc_1 - ry_1)} \tag{23}$$

$$D = \frac{2zc_1 + 2c_2u\gamma^2y_1 + xu^2\gamma^2c_1 - 2y_2u\gamma^2c_1 - 2\chi y_1 - ru^2\gamma^2y_1}{\omega^2(xc_1 - ry_1)} \tag{24}$$

The circle is given in the (s, t)-plane by

$$(s - C)^2 + t^2 = C^2 + D \tag{25}$$

□

Remark 3.1. This is for pseudo walls, which now also implies the theorem for walls.

With the notation above, we will state but not prove

Lemma 3.1. *If $x \neq 0$, then*

$$D = \frac{-u\gamma^2(2y_2 - ux) + 2z}{\omega^2 x} - \frac{2y_1}{x}C \tag{26}$$

You can also see, it will be independent of the radius if $x = 0$

Lemma 3.2. *If $x = 0$ and $y_1 > 0$, then*

$$C = \frac{z - \gamma^2uy_2}{\omega^2y_1} \tag{27}$$

Proof. C reduces to

$$C = \frac{0 \cdot \chi - rz - u\gamma^2(0 \cdot c_2 - ry_2)}{\omega^2(0 \cdot c_1 - ry_1)} \tag{28}$$

$$= \frac{z - u\gamma^2 y_2}{\omega^2 y_1} \tag{29}$$

□

Remark 3.2. If $x > 0$, the radius is

$$R = \sqrt{C^2 + D} = \sqrt{(C - \frac{y_1}{x})^2 - F} \tag{30}$$

$$F = -\frac{\gamma^2}{\omega^2}(u - \frac{y_2}{x})^2 + \frac{1}{x^2\omega^2}(y_1^2\omega^2 + y_2^2\gamma^2 - 2xz) \tag{31}$$

Theorem 3.1. *Suppose v has positive discriminant. The walls are nested*

Proof. If $x = 0$, it follows immediately since the radii are independent of w. We consider the function

$$\frac{d}{dC}(C \pm R(C)) = (1 \pm \frac{\frac{d}{dC}((C - y_1/x)^2 - F)}{2\sqrt{R}}) \tag{32}$$

$$= 1 \pm \frac{2(C - y_1/x)}{2\sqrt{(C - y_1/x)^2 - F}} \tag{33}$$

$$= 1 \pm \frac{(C - y_1/x)}{R} \tag{34}$$

By the Hodge index theorem, we have $F \geq 0$ for all u, therefore

$$R = \sqrt{(C - y_1/x)^2 - F} \leq |C - y_1/x| \tag{35}$$

Now we have to do some case distinction. Let $C - y_1/x > 0$

$$\frac{d}{dC}(C + R(C)) \geq 1 + 1 > 0 \tag{36}$$

$$\frac{d}{dC}(C - R(C)) \leq 1 - 1 = 0 \tag{37}$$

So in this case, we get that for increasing C the interval $[C - R, C + R]$ keeps getting larger, so they have to be nested. □