Renormalizing a Scalar-Fermion Yukawa Model

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This is a calculation I had to do from a second course in quantum field theory PHYC90057. It beautifully demonstrates a lot of the machinery used in perturbative quantum field theory[1] and as far I know is nowhere on the internet. The exposition is verbose throughout, with the hope that other graduate students find it helpful.

1 Introduction

Consider the bare lagrangian describing a two-component massless spinor ψ and two complex scalar fields ϕ and S

$$\mathcal{L} = (\partial_\mu \phi)^\dagger \partial^\mu \phi + \psi^\dagger i\overline{\sigma}^\mu \partial_\mu \psi + S^* S + [\lambda_1 S \phi^2 + i\lambda_2 \psi^T \sigma^2 \psi \phi + \textbf{h.c.}] - m^2 |\phi|^2,$$

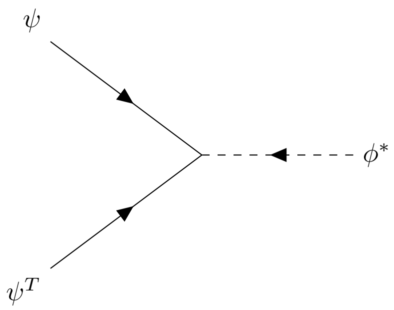
where $(\overline{\sigma})^\mu := (I, -\sigma^i)$ and the couplings λ_1, λ_2 are taken to be real. Treating S and S^* as independent fields, the Euler–Lagrange equations $\partial\mathcal{L}/\partial S = 0$ and $\partial\mathcal{L}/\partial S^* = 0$ give the equations of motion

$$0 \;=\; \frac{\partial \mathcal{L}}{\partial S^*} \;=\; S + \lambda_1 (\phi^*)^2 \quad \text{ and } \quad 0 \;=\; \frac{\partial \mathcal{L}}{\partial S} \;=\; S^* + \lambda_1 \phi^2.$$

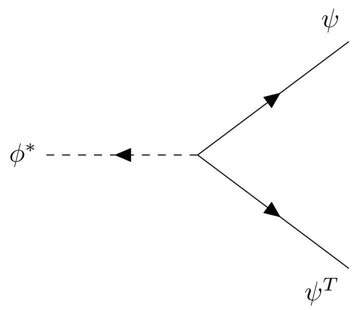
Solving the equations for S and S^*, we find the on-shell solutions are given by $S = -\lambda_1(\phi^*)^2$ and $S^* = -\lambda_1 \phi^2$. The complex scalar fields S and S^* can then be integrated out of $\mathcal{L}$ by substituting our on-shell expressions back into the Lagrangian

$$\mathcal{L} = |\partial\phi|^2 + \psi^\dagger (i\overline{\sigma} \cdot \partial)\psi + i\lambda_2 \psi^T \sigma^2 \psi \phi - i\lambda_2 \psi^\dagger \sigma^2 \psi^* \phi^* - m^2 |\phi|^2 - \lambda_1^2 |\phi|^4.$$

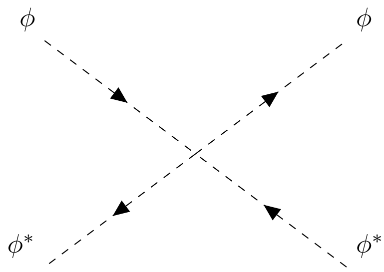
The above is an effective Lagrangian in the fields ϕ and ψ with now an additional quartic self-interaction term for ϕ. From $\mathcal{L}$ we can determine the interaction vertices for our theory:



$= -2\lambda_2 \sigma^2,$



$= 2\lambda_2 \sigma^2$



$= -4i\lambda_1^2,$

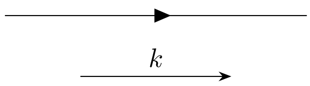
where we have taken into account the symmetry factor of the quartic self-interaction coming from the connected four point scalar function. Observe at first order there are 4 possible contractions of $|i\rangle$ and $\langle f|$

$$\langle p, \tilde{p}| T\phi\phi^*\phi\phi^* |k, \tilde{k}\rangle = \langle p, \tilde{p}|\phi\phi^*\phi\phi^*|k, \tilde{k}\rangle + \langle p, \tilde{p}|\phi\phi^*\phi\phi^*|k, \tilde{k}\rangle + \langle p, \tilde{p}|\phi\phi^*\phi\phi^*|k, \tilde{k}\rangle + \langle p, \tilde{p}|\phi\phi^*\phi\phi^*|k, \tilde{k}\rangle.$$

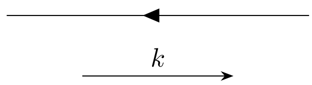
[1] whilst also suppressing superficial complexities like spinor indices. The indices can be done away with as long as the orientations of the Feynman diagrams given are adhered to. This is why our complex scalar field has a directed propagator.

2 The One-Loop Correction of the Two-Point Function for ϕ

We begin by introducing the momentum-space representation for the fermion propagator, given by



$= \frac{ik_\mu \sigma^\mu}{k^2}$ and



$= \frac{ik_\mu \overline{\sigma}^\mu}{k^2}$.

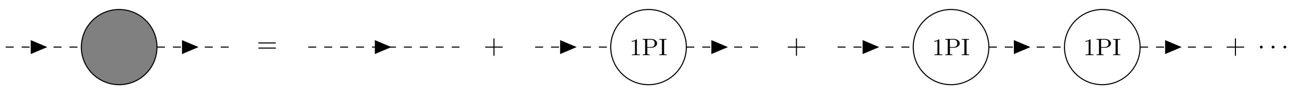
Before computing corrections to the propagator at one loop, we recount some basic theory. Recall that a 1-particle irreducible (1PI) diagram is by definition any diagram that cannot be split in two by removing a single line. Diagrammatically, the circle with 1PI in the centre represents the sum over all 1-PI two-point diagrams. Algebraically we denote it by the expression $-i\Pi_\phi(p^2)$ called the 1PI *amplitude* of ϕ where $\Pi(p^2)$ is the *self energy of the scalar field* ϕ. The 2-point Greens function in momentum space denoted $D_F(p)$ is given by summing over all connected two point diagrams. However observe that a connected two point Feynman diagram can be decomposed recursively into a bare propagator plus a bare propagator concatenated with a 1PI diagram connected to an arbitrary connected two point diagram. This recursion is stated algebraically as

$$D_F(p) = D_0(p) + D_0(p)[-i\Pi(p^2)]D_F(p),$$

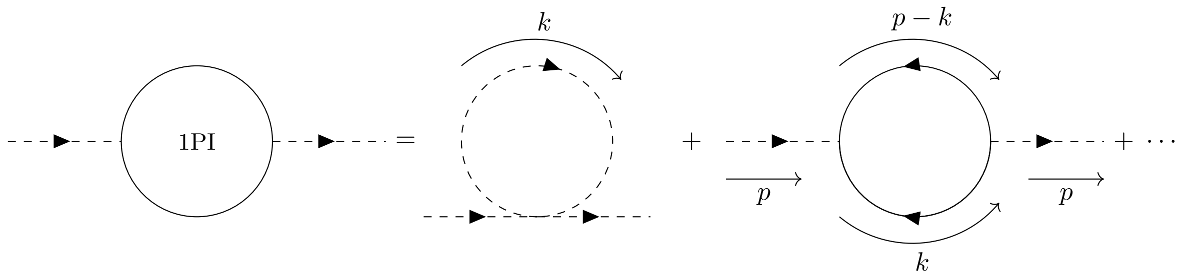
where $D_0(p)$ is the *bare propagator*. By repeated substitution the recursion has the form of a geometric series which can then be resummed as

$$D_F(p) = \frac{i}{p^2 - (m^2 + \Pi_r(p^2)) + i\varepsilon},$$

where $\Pi_r(p^2)$ denotes the *renormalized self-energy* of the scalar field ϕ.[2]The recursion above also implies that $D_F(p)$ can be expressed as the following sum of diagrams:



Using our interaction vertices from the previous page we can deduce the form of the diagrams that appear in the expansion of the 1PI amplitude of ϕ to order $O(\lambda_1^2, \lambda_2^2)$, they are



Now applying the standard Feynman rules to the above diagrams respectively gives

$$-i\Pi_\phi = -4i\lambda_1^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} = 4\lambda_1^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2},$$

$$-i\Pi_\psi(p^2) = \frac{1}{2}(-2\lambda_2)(2\lambda_2) \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left(\frac{\sigma^2 i(p-k)\cdot \overline{\sigma}^T}{(p-k)^2} \sigma^2 \frac{i(k\cdot\overline{\sigma}}{k^2} \right) = 2\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \frac{(p-k)_\mu k_\nu}{(p-k)^2 k^2} \operatorname{Tr}\left(\sigma^2 (\overline{\sigma}^T)^\mu \sigma^2 \overline{\sigma}^\nu\right).$$

Since $(\overline{\sigma}^T)^\mu = (I, -\sigma^i)$ it follows that the identity $\sigma^2(\overline{\sigma}^T)^\mu \sigma^2 = \sigma^\mu$ holds. Then combining this with the standard QFT trace identity of the product of two Pauli matrices yields

$$\operatorname{Tr}(\sigma^2(\overline{\sigma}^T)^\mu \sigma^2 \overline{\sigma}^\nu) = \operatorname{Tr}(\sigma^\mu \sigma^\nu) = 2g^{\mu\nu}.$$

[2]It is worth pointing out that the pole in the denominator (when compared with the pole in the formula of $D_F(p)$ given by the Källen-Lehmann spectral representation) relates physical mass squared of ϕ to the bare mass by $m^2_{\text{phys}} = m^2 + \Pi_r(m^2)$.

The above identity can be applied and the Minkowski product $(k-p)\cdot k := (k-p)_\mu k_\nu g^{\mu\nu}$ can be expanded to simplify expression

$$-i\Pi_\psi(p^2) = -4\lambda_2^2 \int \frac{d^4k}{(2\pi)^4}\frac{(k-p)\cdot k}{(p-k)^2k^2}.$$

Adding the contribution of each diagrams gives the 1PI amplitude of ϕ with $-i\Pi(p^2) = -i\Pi_\phi - i\Pi_\psi(p^2)$ to be

$$-i\Pi(p^2) = 4\lambda_1^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2} - 4\lambda_2^2 \int \frac{d^4k}{(2\pi)^4}\frac{(k-p)\cdot k}{(p-k)^2k^2}.$$

Note from the above equation it can be seen that as the loop 4-momentum k goes to the UV there is a quadratic divergence

$$-i\Pi(p^2) \to 4(\lambda_1^2 - \lambda_2^2)\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2},$$

unless the couplings are equal in which case there is a UV fixed point. To calculate the one loop correction we make use the following three standard techniques in sequence; Feynman parameters, Wick rotation and dimensional regularization. To begin, we introduce the technique of Feynman parameters which involves exploiting integral identities of the form (or similar to)

$$\frac{1}{A_1A_2} = \int_0^1 dx \frac{1}{(A_1 + (A_2 - A_1)x)^2}.$$

The product of factors in the denominator can then be combined in a way where they form a single quadratic expression raised to the second power. Then after completing the square and changing variables the integral over the momentum variable can now be done however there will also be auxiliary variables that can be integrated out. Therefore for the current integral at hand we apply the integral identity above to $-i\Pi_\psi(p^2)$ and complete the square to obtain

$$\begin{aligned}\frac{1}{k^2(p-k)^2} &= \int_0^1 dx \frac{1}{(k^2 + ((p-k)^2 - k^2)x)^2}\\ &= \int_0^1 dx \frac{1}{(k^2 + p^2x - 2pkx)^2}\\ &= \int_0^1 dx \frac{1}{((k-px)^2 - p^2x(x-1))^2}.\end{aligned}$$

Define $\Delta := p^2x(x-1)$ and $\ell := k - px$ then $d^4\ell = d^4k$, the numerator simplifies as

$$\begin{aligned}(k-p)\cdot k &= (\ell + p(x-1))\cdot(\ell + px)\\ &= \ell^2 + \Delta + O(\ell).\end{aligned}$$

Noting that odd integrals vanish our full expression simplifies to

$$\begin{aligned}-i\Pi_\psi(p^2) &= -4\lambda_2^2 \int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{(k-p)\cdot k}{((k-px)^2 - p^2x(x-1))^2}\\ &= -4\lambda_2^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\left[\frac{\ell^2}{(\ell^2-\Delta)^2} + \frac{\Delta}{(\ell^2-\Delta)^2}\right],\end{aligned}$$

where we are left with a quadratic divergence and logarithmic divergence respectively. Currently the loop integrals above are written with contractions using a Lorentz signature. However if they were in Euclidean space we could do them in spherical polar coordinates in a relatively straight-forward way. To progress further we introduce our second technique, *Wick rotation.* A Wick rotation will transform the Lorentz invariance into rotational invariance. To perform a Wick rotation we define Euclidean 4-momentum variables $\ell_E^0 := -i\ell^0$ and $\ell_E^i = \ell^i$ this implies $\ell^2 = -\ell_E^2$ and $d^4\ell = id^4\ell_E$. Here we have analytically continued one of the variables into the complex plane. Wick rotating, our expression becomes

$$-i\Pi_\psi(p^2) = -4i\lambda_2^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \left[\frac{-\ell_E^2}{(\ell_E^2+\Delta)^2} + \frac{\Delta}{(\ell_E^2+\Delta)^2} \right].$$

Unfortunately, this integral is still divergent in the UV regime. One possible way to get around this divergence would be to introduce a cut-off, and that way we could still obtain some approximate answer. Unfortunately, it turns out whatever answer we would obtain would violate the Ward identity and introduce a mass that is proportional to the cut-off. Therefore in order to study the behaviour of this loop integral in a way which respects the Ward identity we use a different technique known as *dimensional regularisation*. The idea of dimensional regularisation is to replace dimension 4 with dimension d, and interpret $d=4$ as a divergence. Then we can use perturbation theory to extract the leading order contributions for the divergent integral at $d=4$ by working in dimension $d = 4-\varepsilon$.

Consider the same loop integral as above but now in d-dimensions (where $d \neq 4$). The d-dimensional closed form of loop integrals of this type are a standard results in QFT and hence we use them without giving a derivation. The first loop integral is solved using the identity

$$\int \frac{d^d\ell_E}{(2\pi)^d} \frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1},$$

and for the second loop integral we use the identity

$$\int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2+\Delta)^2} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}.$$

There is a subtlety when working in an arbitrary integer dimension, namely our couplings are no longer dimensionless. Doing some elementary dimensional analysis on our Lagrangian we find that our couplings rescale according to $\lambda_2 \mapsto \mu^{\frac{4-d}{2}}\lambda_2$ where μ is an arbitrary parameter we introduce of mass dimension one. Therefore for $d \neq 4$ our 1PI amplitude has the form

$$-i\Pi_\psi(p^2) = -4i\lambda_2^2\mu^{4-d} \int_0^1 dx \left[\frac{-1}{(4\pi)^d} \frac{d}{2} \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-\frac{d}{2}}} + \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} \right].$$

Substituting $d = 4-\varepsilon$, and applying elementary complex analysis we obtain the expansion for each term:

- $\Gamma(1-\frac{d}{2}) = \Gamma(-1+\frac{\varepsilon}{2}) \approx -\frac{2}{\varepsilon} + \gamma_E - 1$;
- $\Gamma(2-\frac{d}{2}) = \Gamma(\frac{\varepsilon}{2}) \approx \frac{2}{\varepsilon} - \gamma_E$;
- $\mu^{4-d} = \mu^\varepsilon = (\mu^2)^{\frac{\varepsilon}{2}} \approx 1 + \frac{\varepsilon}{2}\log(\mu^2)$;
- $(4\pi)^{-\frac{d}{2}} = (4\pi)^{-2+\frac{\varepsilon}{2}} \approx \frac{1}{(4\pi)^2}\left(1+\frac{\varepsilon}{2}\log(4\pi)\right)$;
- $\Delta^{\frac{d}{2}-1} = \Delta^{1-\frac{\epsilon}{2}} \approx \Delta(1-\frac{\epsilon}{2}\log(\Delta))$;
- $\Delta^{\frac{d}{2}-2} = \Delta^{-\frac{\varepsilon}{2}} \approx 1 - \frac{\varepsilon}{2}\log(\Delta)$.

Define the constant $\tilde{\mu}^2 := 4\pi e^{-\gamma_E}\mu^2$, expanding both expressions up to order $O(\varepsilon)$ yields

$$\begin{aligned}
\frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-\frac{d}{2}}} &= \frac{\Delta}{16\pi^2}\left(-\frac{2}{\varepsilon} + \gamma_E + -1\right)\left(1-\frac{\varepsilon}{2}\log(\Delta)\right)\left(1+\frac{\varepsilon}{2}\log(\mu^2)\right)\left(1+\frac{\varepsilon}{2}\log(4\pi)\right) \\
&= \frac{\Delta}{16\pi^2}\left[-\frac{2}{\varepsilon} + \gamma_E - 1 + \log(\Delta) - \log(\mu^2) - \log(4\pi)\right] \\
&= -\frac{\Delta}{16\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{\Delta}\right) + 1\right],
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}} &= \frac{1}{16\pi^2}\left(1-\frac{\varepsilon}{2}\log(\Delta)\right)\left(\frac{2}{\varepsilon}-\gamma_E\right)\left(1+\frac{\varepsilon}{2}\log(4\pi)\right)\left(1+\frac{\varepsilon}{2}\log(\mu^2)\right) \\
&= \frac{1}{16\pi^2}\left(\frac{2}{\varepsilon}-\gamma_E-\log(\Delta)\right)\left(1+\frac{\varepsilon}{2}\log(4\pi)+\frac{\varepsilon}{2}\log(\mu^2)\right) \\
&= \frac{1}{16\pi^2}\left(\frac{2}{\varepsilon}+\log(4\pi)+\log(\mu^2)-\gamma_E-\log(\Delta)\right) \\
&= \frac{\Delta}{16\pi^2}\left(\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)\right).
\end{aligned}$$

Combining everything together up to order ε yields our 1-loop contribution from the second Feynamn diagram:

$$\begin{aligned}
-i\Pi_\psi(p^2) &= -4i\lambda_2^2\int_0^1 dx\left[-\frac{\Delta}{16\pi^2}\left(-2+\frac{\varepsilon}{2}\right)\left(\frac{2}{\varepsilon}+\log(\frac{\tilde{\mu}^2}{\Delta})+1\right)+\frac{\Delta}{16\pi^2}\left(\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)\right)\right] \\
&= -\frac{3i\lambda_2^2}{4\pi^2}\int_0^1 dx\Delta\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)+\frac{1}{3}+O(\varepsilon)\right].
\end{aligned}$$

Recall that we also had to evaluate the loop integral associated to correction from the first Feynman diagram

$$-i\Pi_\phi^2 = 4\lambda_1^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}.$$

Performing a Wick rotation, we set $k_E^0 := -ik^0$ and $k^2 = -k_E^2$ which yields

$$-i\Pi_\phi^2 = -4i\lambda_1^2\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2}.$$

Dimensional regularizing, and making use of d-dimensional loop integral identity, gives

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k_E^2+m^2)} = \frac{1}{(4\pi)^{\frac{d}{2}}}\left(\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}}\right).$$

Using the expansion derived before we expand each term out to first order

$$\begin{aligned}
-i\Pi_\phi &= -\frac{4i\lambda_1^2\mu^{4-d}}{(4\pi)^{\frac{d}{2}}}\left(\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}}\right) \\
&= -4i\lambda_1^2\left(m^2(1-\frac{\varepsilon}{2}\log(m^2))\right)\left(-\frac{2}{\varepsilon}+\gamma_E+1\right)\frac{1}{(4\pi)^2}\left(1+\frac{\varepsilon}{2}\log(4\pi)\right)\left(1+\frac{\varepsilon}{2}\log(\mu^2)\right) \\
&= -\frac{i\lambda_1^2m^2}{4\pi^2}\left(1-\frac{\varepsilon}{2}\log(m^2)\right)\left(-\frac{2}{\varepsilon}+\gamma_E-1\right)\left(1+\frac{\varepsilon}{2}\log(4\pi)\right)\left(1+\frac{\varepsilon}{2}\log(\mu^2)\right) \\
&= -\frac{i\lambda_1^2m^2}{4\pi^2}\left[-\frac{2}{\varepsilon}+\log(m^2)+\gamma_E-1-\log(4\pi)-\log(\mu^2)\right] \\
&= \frac{i\lambda_1^2m^2}{4\pi^2}\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{m^2}\right)+1\right].
\end{aligned}$$

Putting everything together we find that the one-loop correction to the two point function for ϕ is given by

$$i\Pi(p^2) = \frac{i\lambda_1^2m^2}{4\pi^2}\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{m^2}\right)+1\right]-\frac{3i\lambda_2^2}{4\pi^2}\int_0^1 dx\Delta\left[\frac{2}{\varepsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)+\frac{1}{3}\right]+O(\varepsilon).$$

3 A Renormalized Perturbation Theory for $\mathcal{L}$ in the $\overline{\text{MS}}$ Scheme

In this section we give a full account of how to construct a renormalized perturbation theory for $\mathcal{L}$ in the *modified minimal subtraction* scheme $\overline{\text{MS}}$. Recall that in the MS scheme the finite parts of the counterterms are chosen to be zero and in the $\overline{\text{MS}}$ scheme the universal constants the appear in the regularization (which are $\log(4\pi) + \gamma_E$) are subtracted off[3].

From now we denote the bare fields and masses by ψ_0, ϕ_0 and m_0 respectively. Recall the bare Lagrangian after integrating out S was given by

$$\mathcal{L}_0 = |\partial\phi_0|^2 + \psi_0^\dagger(i\overline{\sigma}\cdot\partial)\psi_0 + i\lambda_2\psi_0^T\sigma^2\psi_0\phi_0 - i\lambda_2\psi_0^\dagger\sigma^2\psi_0^*\phi_0^* - m_0^2|\phi_0|^2 - \lambda_1^2|\phi_0|^4.$$

We denote the renormalized fields and bare masses by ψ, ϕ and m respectively. We introduce the following wavefunction renormalizations

$$\psi_0 := Z_\psi^{\frac{1}{2}}\psi \text{ and } \phi_0 := Z_\phi^{\frac{1}{2}}\phi.$$

Our renormalized the infinities appearing in our bare theory we will introduce a renormalized Lagrangian $\mathcal{L} = \mathcal{L}_0 - \mathcal{L}_{\text{ct}}$ where $\mathcal{L}_{\text{ct}}$ is the counterterm lagrangian which will be carefully deduced and which will cancel with the divergences. Substituting the definition of the bare quantities into our bare Lagrangian $\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\text{c.t.}}$ we have

$$\begin{aligned}\mathcal{L}_0 &= |\partial\phi_0|^2 + \psi_0^\dagger(i\overline{\sigma}\cdot\partial)\psi_0 + i\lambda_{2,0}\psi_0^T\sigma^2\psi_0\phi_0 - i\lambda_{2,0}\psi_0^\dagger\sigma^2\psi_0^*\phi_0^* - m_0^2|\phi_0|^2 - \lambda_{1,0}^2|\phi_0|^4 \\ &= Z_\phi|\partial\phi|^2 + Z_\psi\psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + [i\lambda_{2,0}Z_\phi^{\frac{1}{2}}Z_\psi\psi^T\sigma^2\psi\phi + \textbf{h.c.}] - m_0^2Z_\phi|\phi_0|^2 - \lambda_{1,0}^2Z_\phi^2|\phi_0|^4.\end{aligned}$$

To deduce the additional counterterms, first observe that $\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\text{c.t.}}$, this decomposition therefore suggests the relations

$$Z_\phi = 1 + \delta_\phi, \quad Z_\psi = 1 + \delta_\psi,$$

$$\lambda_{1,0}^2Z_\phi^2 = \lambda_1^2 + \delta_1, \quad \lambda_{2,0}Z_\phi^{\frac{1}{2}}Z_\psi = \lambda_2 + \delta \quad \text{and} \quad m_0^2Z_\phi = m^2 + \delta_m.$$

Since $\mathcal{L}_{\text{c.t.}} = \mathcal{L}_0 - \mathcal{L}$ we can rearrange to deduce the respective definitions of the counterterms:

$$\delta_\phi := Z_\phi - 1, \quad \delta_\psi := Z_\psi - 1,$$

$$\delta_1 := \lambda_1^2 - \lambda_{1,0}^2Z_\phi^2 = \lambda_1^2 - \lambda_{1,0}^2Z_1, \quad \delta_2 := \lambda_{2,0}Z_\phi^{\frac{1}{2}}Z_\psi - \lambda_2\delta = \lambda_{2,0}Z_2 - \lambda_2 \quad \text{and} \quad \delta_m := m_0^2Z_\phi - m^2,$$

with $Z_1 := Z_\phi^2$ and $Z_2 := Z_\phi^{\frac{1}{2}}Z_\psi$. The above relations can then be substituted into the bare Lagrangian which gives

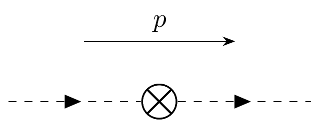
$$\begin{aligned}\mathcal{L}_0 &= (1+\delta_\phi)|\partial\phi|^2 + (1+\delta_\psi)\psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + [i(\lambda_2+\delta_2)\psi^\dagger\sigma^2\psi\phi + \textbf{h.c.}] - (m^2+\delta_m)|\phi|^2 - (\lambda_1^2+\delta_1)|\phi|^4 \\ &= \mathcal{L} + \delta_\phi|\partial\phi|^2 + \delta_\psi\psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + [i\delta_2\psi^\dagger\sigma^2\psi\phi + \textbf{h.c.}] - \delta_m|\phi|^2 - \delta_1|\phi|^4.\end{aligned}$$

Therefore the counterterm Lagrangian is given by

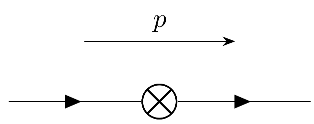
$$\mathcal{L}_{\text{c.t.}} = \delta_\phi|\partial\phi|^2 + \delta_\psi\psi^\dagger(i\overline{\sigma}\cdot\partial)\psi + [i\delta_2\psi^\dagger\sigma^2\psi\phi + \textbf{h.c.}] - \delta_m|\phi|^2 - \delta_1|\phi|^4.$$

From the counterterm Lagrangian the Feynman diagram and rules for the counterterm insertions can be read off as

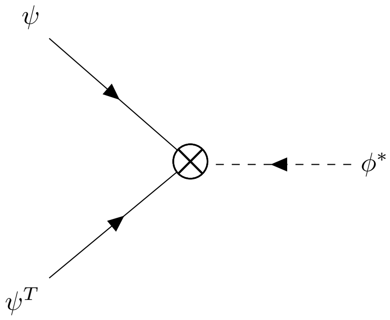
[3] The advantage of using the $\overline{\text{MS}}$ scheme is that it is makes it easier to deduce the counterterms, additionally the 1-loop expressions also have a simpler form. The disadvantage is that it is not strictly true that the renormalized mass is the physical mass. An on-shell subtraction scheme can be used alternatively, however this scheme has the opposite advantages and disadvantages of the former.



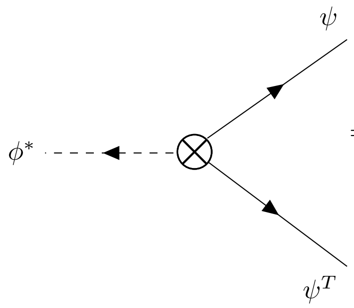
$= i(p^2\delta_\phi - \delta_m),$ and



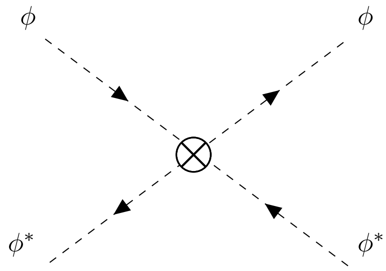
$= i\delta_\psi \sigma \cdot p.$



$= -2\delta_2\sigma^2,$



$= 2\delta_2\sigma^2$



$= -4i\delta_1$

We now need to compute the radiative corrections to each of the amputated vertices adding the counterterm insertion.

Firstly the three-point function $\delta_2 = 0$, as it's clear that neither $\psi\psi^T \mapsto \phi^*$ nor $\phi^* \mapsto \psi\psi^T$ have radiative corrections at one-loop. This is because loop cannot be formed in such a way with the diagrams that keeps the orientations of the edges consistent.

Next we derive counterterms for the scalar 2-point function. In the previous section we computed the bosonic self energy however now we must now also take into account the Feynman diagram associated to the δ_ϕ and δ_m counterterm insertions. Using our previous result and expanding

$$
\begin{aligned}
-i\Pi(p^2) &= -i\Pi_\phi - i\Pi_\psi(p^2) + i(p^2\delta_\phi - \delta_m) \\
&= \frac{i\lambda_1^2 m^2}{4\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{m^2}\right) + 1\right] - \frac{3i\lambda_2^2}{4\pi^2}\int_0^1 dx\Delta\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{\Delta}\right) + \frac{1}{3}\right] + i(p^2\delta_\phi - \delta_m),
\end{aligned}
$$

since our counterterms should cancel these loop corrections, we set the respective counterterms equal to

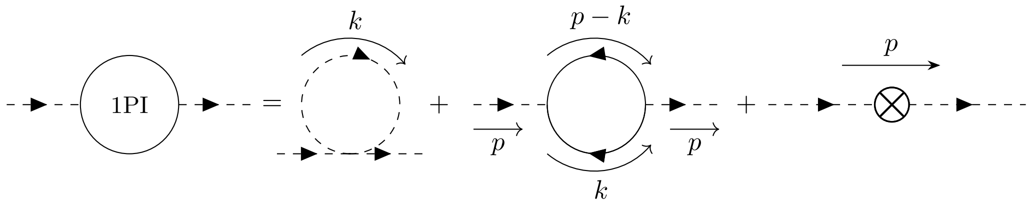
$$
-i\delta_m = i\Pi_\phi \quad \text{and} \quad -i\delta_\phi = i\lim_{p^2 \longrightarrow 0}\frac{\partial \Pi_\psi}{\partial p^2}.
$$

Therefore

$$
\delta_m = \Pi_\phi = \frac{\lambda_1^2 m^2}{4\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{m^2}\right) + 1\right],
$$

now applying the $\overline{\text{MS}}$ scheme, the counterterm δ_m is redefined using the above equation but with finite parts dropped and with universal constants included

$$
\delta_m := \frac{\lambda_1^2 m^2}{2\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}).
$$



Similarly we compute

$$\begin{aligned}\delta_\phi &= \lim_{p^2\to 0}\frac{\partial}{\partial p^2}\left(\frac{-3\lambda_2^2 i}{4\pi^2}\int_0^1 dx\ \Delta\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)+\frac{1}{3}-\log(\Delta)\right]\right)\\ &= \frac{-3\lambda_2^2 i}{4\pi^2}\int_0^1 dx\ x(x-1)\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)-1-\lim_{p^2\to 0}\frac{\partial}{\partial p^2}\left(p^2\log(p^2x(x-1))\right)\right]\\ &= \frac{-3\lambda_2^2 i}{4\pi^2}\int_0^1 dx\ x(x-1)\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)-\frac{2}{3}-\lim_{p^2\to 0}\log(\Delta)\right].\end{aligned}$$

As $p^2 \to 0$, the term $\log(\Delta) \to -\infty$ and so there is an IR divergence. To regulate this divergence we now introduce a fictitious fermion mass term M_ψ such that $M_\psi^2 << \mu^2$. The Δ above can then be replaced with the IR regulated Δ given by

$$\Delta := \Delta_{\text{IR}} = p^2x(x-1)+M_\psi^2.$$

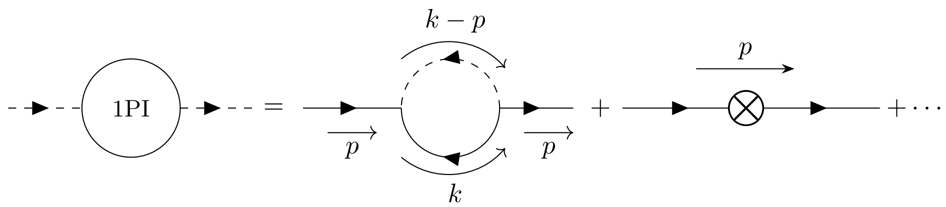
Now we can take $p^2 \to 0$ to obtain a finite expression that isolates the ultraviolet pole and keeps the IR regulator

$$\begin{aligned}\delta_\phi^{\text{reg}} &= \frac{-3\lambda_2^2}{4\pi^2}\int_0^1 dx\ x(x-1)\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)-\frac{2}{3}-\log(M_\psi^2)\right]\\ &= \frac{\lambda_2^2}{8\pi^2}\left[\frac{2}{\varepsilon}+\log(\tilde{\mu}^2)-\frac{2}{3}-\log(M_\psi^2)\right],\ \text{as}\ \int_0^1 x(x-1)dx=\frac{1}{6}.\end{aligned}$$

Applying the $\overline{\text{MS}}$ scheme the corresponding counterterm δ_ϕ above is redefined dropping the finite part and adding in the universal constants

$$\delta_\phi := \frac{\lambda_2^2}{4\pi^2\varepsilon}+\log(4\pi e^{-\gamma_E}).$$

Next to determine is the δ_ψ counterterm. The fermion self energy at 1-loop is given by which mathematically



corresponds to $-i\Sigma(p) = -i\Sigma_1(p) + i\delta_\psi \sigma\cdot p$. In order to determine δ_ψ, we must compute the contribution to the 1-loop amplitude given by $-i\Sigma_1(p)$. Using the Feynman rules and noting that the sign coming from the fermion interchange in $\psi^\dagger\sigma^2\psi^*\psi^T\sigma^2\psi$, yields

$$\begin{aligned}-i\Sigma_1(p) &= 4\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\frac{(i)^2k_\mu\sigma^2\overline{\sigma}^\mu\sigma^2}{k^2[(k-p)^2-m^2]}\\ &= -4\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\frac{k_\mu\sigma^\mu}{k^2[(k-p)^2-m^2]}\quad \text{since } \sigma^2\overline{\sigma}^\mu\sigma^2=\sigma^\mu.\end{aligned}$$

Using Feynman parameters where $A = k^2$ and $B = (k-p)^2 - m^2$ we obtain

$$\begin{aligned} A + (B-A)x &= k^2 + ((k-p)^2 - k^2 - m^2)x \\ &= k^2 + xp^2 - 2xkp - m^2x \\ &= (k-xp)^2 - x^2p^2 + xp^2 - m^2x. \end{aligned}$$

Let $\Delta := x^2p(1-x) + m^2x$, and let $\ell = k - xp$ then $d^4\ell = d^4k$, substituting this into our expression gives

$$-i\Sigma_1(p) = -4\lambda_2^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell + xp)_\mu \sigma^\mu}{(\ell^2 - \Delta)^2} = -4\lambda_2^2 (\sigma \cdot p) \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{x}{(\ell^2 - \Delta)^2},$$

where the integral with ℓ to the odd power is odd and hence vanishes. Performing a Wick rotation: set $\ell_E^0 := -i\ell^0$ and set $\ell_E^i := \ell^i$ then $d^4\ell = id^4\ell_E$ and $\ell_E^2 = -\ell^2$, so

$$-i\Sigma_1(p) = -4i\lambda_2^2(\sigma \cdot p) \int_0^1 dx\, x \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{(\ell_E^2 + \Delta)^2}.$$

Applying dimensional regularization, where $d = 4 - \varepsilon$ and using the standard identities, expanding to order ε gives

$$-i\Sigma_1(p) = \frac{-i\lambda_2^2(\sigma \cdot p)}{4\pi^2} \int_0^1 dx\, x \left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}}{\Delta}\right) + O(\varepsilon)\right] = \frac{-i\lambda_2^2(\sigma \cdot p)}{8\pi^2} \left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}}{\Delta}\right) + O(\varepsilon)\right],$$

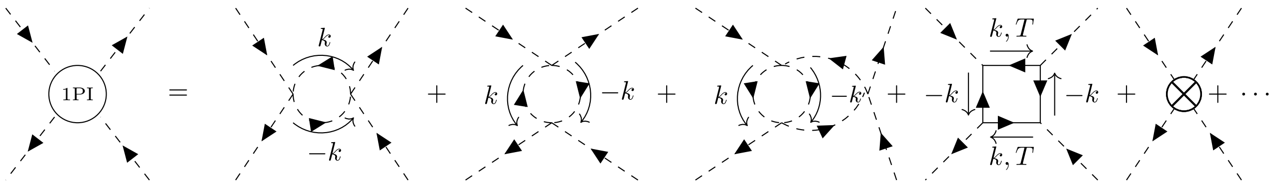
hence our counterterm for ψ should be equal to

$$\delta_\psi = \frac{\lambda_2^2}{8\pi^2} \left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}}{\Delta}\right) + O(\varepsilon)\right].$$

Again, applying the $\overline{\text{MS}}$ scheme the corresponding counterterm δ_ψ above is redefined dropping the finite part and adding in the universal constants

$$\delta_\psi := \frac{\lambda_2^2}{4\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}).$$

Next we compute the δ_1 counterterm by considering a 1-loop correction to the 4-point scalar interaction. Setting the external momenta to zero, we have at 1-loop that:



$\Gamma^{(4)}_{\text{1-loop}}$ Λ_s Λ_t Λ_u Λ_ψ $4i\,\delta_1$

$$i\Gamma^{(4)}_{\text{1-loop}} = i\Lambda_s + i\Lambda_t + i\Lambda_u + i\Lambda_\psi + 4i\delta_1.$$

Computing the $i\Lambda_\psi$ contribution using Feynman rules (and taking into account the sign from fermion interchange) yields

$$\begin{aligned} i\Lambda_\psi &= (-1)(-2\lambda_2)^2(2\lambda_2)^2 \int \frac{d^4k}{(2\pi)^4} i^4 \, \mathrm{Tr}\big(\sigma^2[\overline{\sigma}^\mu]^T \sigma^2 \overline{\sigma}^\nu \sigma^2 [\overline{\sigma}^\rho]^T \sigma^\lambda\big) \frac{k_\mu(-k_\nu)\, k_\rho(-k_\lambda)}{k^8} \\ &= -16\lambda_2^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\big(\sigma^2[\overline{\sigma}^\mu]^T \sigma^2 \overline{\sigma}^\nu \sigma^2 [\overline{\sigma}^\rho]^T \sigma^\lambda\big) \frac{k_\mu k_\nu k_\rho k_\lambda}{k^8}. \end{aligned}$$

The expression using the trace simplifies using the fact that $\sigma^2[\overline{\sigma}^\mu]^T\sigma^2 = \sigma^\mu$ together with an identity for the trace

$$\begin{aligned}\text{Tr}\big(\sigma^2[\overline{\sigma}^\mu]^T\sigma^2\overline{\sigma}^\nu\sigma^2[\overline{\sigma}^\rho]^T\sigma^\lambda\big) &= \text{Tr}\big(\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\lambda\big)\\ &= 2\big[g^{\mu\nu}g^{\lambda\rho} + g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho} - i\,\epsilon^{\mu\nu\lambda\rho}\big].\end{aligned}$$

Using the fact that $k_\mu g^{\mu\nu}k_\nu = k\cdot k = k^2$, the exoression for $i\Lambda_\psi$ simplifies

$$\begin{aligned}i\Lambda_\psi &= -32\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\big(g^{\mu\nu}g^{\lambda\rho} + g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho} - i\,\epsilon^{\mu\nu\lambda\rho}\big)\frac{k_\mu k_\nu k_\rho k_\lambda}{k^8}\\ &= -32\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^4},\end{aligned}$$

Next we introduce a mass regulator M, such that $M^2 << \mu^2$ to regulate the IR divergence

$$i\Lambda_\psi = -32\lambda_2^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2)^2} = -32\lambda_2^2 i\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+M^2)^2},$$

where we Wick rotated, setting $k_E^0 := -ik^0$, $k_E^i := k^i$ so $id^4k_E = d^4k$ and $k^2 = -k_E^2$. From this together with our standard d-dimensional integral identities we can dimensionally regularise this loop integral as follows. Set $d = 4-\varepsilon$, expanding up to first order then yields

$$i\Lambda_\psi = -32i\lambda_2^4(\mu^2)^{4-d}\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{(k^2+M^2)^2} = -\frac{2i\lambda_2^4}{\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{M^2}\right) + O(\varepsilon)\right].$$

Similarly, to compute the $i\Lambda_s$ we apply the Feynman rules to the s-channel diagram to get our loop expression. Performing the calculation just as we did above and expanding to first order we obtain

$$\begin{aligned}i\Lambda_s &= (-4i\lambda_1)^2\int\frac{d^4k}{(2\pi)^4}\frac{(i^2)}{(k^2-m^2)^2}\\ &= \frac{i\lambda_1^4}{\pi^2}\left[\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{m^2}\right) + O(\varepsilon)\right].\end{aligned}$$

Again, applying the Feynman rules to the t and u-channel diagrams, gives expressions of the form

$$i\Lambda_t = i\Lambda_s \quad\text{and}\quad i\Lambda_u = \frac{1}{2}i\Lambda_s.$$

Putting everything together, the 1-loop amplitude for the 4-point function is found to be

$$\begin{aligned}i\Gamma^{(4)}_{\text{1-loop}} &= i\Lambda_s + i\Lambda_t + i\Lambda_u + i\Lambda_\psi + 4i\delta_1\\ &= \frac{5}{2}i\Lambda_s + i\Lambda_\psi + 4i\delta_1.\end{aligned}$$

Hence, our counterterm must be of the form

$$\begin{aligned}\delta_1 &= -\frac{1}{4}\left[\frac{5}{2}\Lambda_s + \Lambda_\psi\right]\\ &= -\frac{1}{4}\left[\frac{5\lambda_1^4}{2\pi^2}\left(\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}}{m^2}\right)\right) - \frac{2\lambda_2^4}{\pi^2}\left(\frac{2}{\varepsilon} + \log\left(\frac{\tilde{\mu}^2}{M^2}\right)\right)\right]\end{aligned}$$

Since we are in the $\overline{\text{MS}}$ scheme the counterterm is then redefined, dropping finite terms, and adding universal constants. Therefore our final counterterm in the renormalized PT theory is given by

$$\delta_1 := -\frac{1}{4\pi^2\varepsilon}\left(5\lambda_1^4 - 4\lambda_2^4\right) + \log(4\pi e^{-\gamma_E}).$$

4 One-Loop β-Functions for $\mathcal{L}$ and RG Flow

To compute the β functions for the parameters in our theory we use the Callman-Symanzik equation,

$$\beta_\lambda = \mu\frac{\partial}{\partial\mu}\left(-\delta_\lambda + \frac{1}{2}\lambda\sum_i \delta_{Z_i}\right) = -2B - \lambda\sum_i A_i$$

where $\delta_\lambda = B\frac{2}{\varepsilon}$ and $\delta_{Z_i} = -A_i\frac{2}{\varepsilon}$. Recall the counterterms we deduced for $\mathcal{L}$ in the previous section:

$$\delta_\psi = \frac{\lambda_2^2}{4\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}), \quad \delta_m = \frac{\lambda_1^2 m^2}{2\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}), \quad \delta_\phi = \frac{\lambda_2^2}{4\pi^2\varepsilon} + \log(4\pi e^{-\gamma_E}),$$

$$\delta_1 = -\frac{1}{4\pi^2\varepsilon}\left(5\lambda_1^4 - 4\lambda_2^4\right) + \log(4\pi e^{-\gamma_E}) \quad \text{and} \quad \delta_2 = 0.$$

The beta function for λ_1^2 is then found by using the formula above with $B = \frac{-1}{8\pi^2}(5\lambda_1^4 - 4\lambda_2^4)$, where we sum over four external legs in the self interaction vertex each contributing $A_\phi = -\frac{\lambda_2^2}{8\pi^2}$ that is

$$\begin{aligned}
\beta_{\lambda_1^2}(\lambda_1^2, \lambda_2^2) &= -2\left(\frac{-1}{8\pi^2}(5\lambda_1^4 - 4\lambda_2^4)\right) - \lambda_1^2\sum_{i=1}^{4}\frac{\lambda_2^2}{8\pi^2} \\
&= \frac{1}{4\pi^2}(5\lambda_1^4 - 4\lambda_2^4) + 4\lambda_1^2\left(\frac{\lambda_2^2}{8\pi^2}\right) \\
&= \frac{1}{4\pi^2}(\lambda_1^4 + 4\lambda_1^2\lambda_2^2 - 4(\lambda_1^2 - \lambda_2^2)).
\end{aligned}$$

To compute the beta function for λ_2 we again use the same formula however this time $B = 0$ as $\delta_2 = 0$,

$$\beta_{\lambda_2}(\lambda_1^2, \lambda_2^2) = -2B - \lambda_2^2\sum_{i=1}^{3} A_i = -\lambda_2^2\left[2\left(-\frac{\lambda_2^2}{8\pi^2}\right) - \frac{\lambda_2^2}{8\pi^2}\right] = \frac{3\lambda_2^3}{8\pi^2}.$$

Having obtained β_{λ_2} for the linear coupling, we now use the chain rule to translate it into the flow of its square

$$\beta_{\lambda_2^2} = \mu\frac{\partial}{\partial\mu}(\lambda_2^2(\mu)) = 2\lambda_2\mu\frac{\partial}{\partial\mu}(\lambda_2(\mu)) = 2\lambda_2\beta_{\lambda_2} = \frac{3\lambda_2^4}{4\pi^2}.$$

Next we analyse the renormalization group flow of the beta functions, in particular we consider whether the condition $\lambda_1^2 = \lambda_2^2$ is stable under renormalization group flow. Let

$$\Delta(\mu) := \lambda_1^2(\mu) - \lambda_2^2(\mu),$$

then if $\Delta = 0$ stays zero for all scales, the condition is said to be preserved. Let $t = \log(\mu)$ the RG equation gives

$$\begin{aligned}
\frac{d\Delta}{dt} &= \frac{d}{dt}(\lambda_1^2 - \lambda_2^2) \\
&= \beta_{\lambda_1^2}(\lambda_1^2, \lambda_2^2) - \beta_{\lambda_2^2}(\lambda_1^2, \lambda_2^2) \\
&= \frac{1}{4\pi^2}(\lambda_1^4 + 4\lambda_1^2\lambda_2^2 + 4(\lambda_1^2 - \lambda_2^2)) - \frac{3\lambda_2^4}{4\pi^2} \\
&= \frac{1}{4\pi^2}((\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + 3\lambda_2^2) + 4(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + \lambda_2^2)) \\
&= \frac{(5\lambda_1^2 + 7\lambda_2^2)}{4\pi^2}\Delta.
\end{aligned}$$

Since $\frac{5\lambda_1^2+7\lambda_2^2}{4\pi^2} > 0$, by the above $\frac{d\Delta}{dt} = 0$ if $\Delta = 0$ for all t, and hence all scales μ. Therefore the condition $\lambda_1 = \lambda_2$ is preserved under renormalization group evolution. Moreover we see that $|\Delta|$ deceases in the IR and increases in the UV.