Quantum Cohomology and the Gromov-Witten Theory of $\mathbb{P}^2$

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November 26, 2024

1 Introduction

We would like to know the answer to the following question:

Question 1.1. *What is the number N_d of degree d rational curves that pass through $(3d-1)$ points in general position on the plane?*

One way to proceed is to get our hands dirty and see if we can spot some kind of pattern. Well, the case when $n = 1$ reduces to a basic question that one first encounters early in school, namely:

Question 1.2. *How many lines pass between two points in the plane?*

The answer if of course 1, so the number of degree 1 rational curves that pass through 2 marked points which are in general position is 1. The next iteration of the question would be for when the degree is 2.

Question 1.3. *How many conics pass between five points in the plane?*

Once again the answer is classically known to be 1 and has been known at least since the time of Apollonius ($\sim$ 200B.C). We can show this by construction as follows: consider points (x_i, y_i) with $1 \leq i \leq 5$ in general position then we have the determinant

$$f(X,Y) = \begin{vmatrix} 1 & X & Y & X^2 & Y^2 & XY \\ 1 & x_1 & y_1 & x_1^2 & y_1^2 & x_1y_1 \\ 1 & x_2 & y_2 & x_2^2 & y_2^2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3^2 & y_3^2 & x_3y_3 \\ 1 & x_4 & y_4 & x_4^2 & y_4^2 & x_4y_4 \\ 1 & x_5 & y_5 & x_5^2 & y_5^2 & x_5y_5 \end{vmatrix}. \tag{1}$$

The determinant here is a polynomial of degree 2 (i.e a conic) in X and Y and substituting in any of our five points we have $f(x_i, y_i) = 0$ as this would yield two identical rows. Therefore five points determine a unique conic in $\mathbb{P}^2$.

Question 1.4. *How many cubics pass between 8 points in the plane?*

This is a much less elementary question and the answer is 12 as computed by Chasles/Steiner in the early 19th century. A sketch of the argument for the proof is given:

Consider a line $f(x,y) + tg(x,y) = 0$ with coordinate t in the space of cubics i.e $\deg(f) = \deg(g) = 3$. The cubics can be seen as a family over $\mathbb{P}^1_{[t]}$. We now compute the Euler characteristic of the total space $\mathcal{S}$ in two ways: By Bezout's theorem f and g intersect at nine points corresponding to nine lines. If $P \in \mathbb{P}^2$ is not one of those nine points then there is one cubic in the family that contains such a point. The total space can be recognized as the blowup of $\mathbb{P}^2$ at 9 points, and hence has Euler characteristic $\chi(\mathcal{S}) = \chi(\mathbb{P}^2 \setminus 9\mathbb{P}^0 \sqcup 9\mathbb{P}^1) = 3 - 9 \cdot 1 + 9 \cdot 2 = 12$. Alternatively, viewing the family fiberwise, the generic fiber is a smooth cubic (torus) and the nodal cubic has $\chi(C_{\text{nodal}}) = 1$ and the Euler characteristic of the family is $\chi(\mathcal{S}) = n_{\text{nodal}}\chi(C_{\text{nodal}}) + n_{\text{smooth}}\chi(C_{\text{smooth}}) = n_{\text{nodal}}$, which gives $n_{\text{nodal}} = 12$.

To summarize we have found that $N_1 = N_2 = 1$, and $N_3 = 12$. It wasn't until the late 19th century that $N_4 = 620$ was computed by Schubert/Zeuthen, and it wasn't until the mid 20th century that we found $N_5 = 87304$. Our results so far are then

$$1, 1, 12, 620, 87304, \ldots$$

Given this data it is not all clear what N_6 should be, and that was the state of things... until Maxim Kontsevich discovered the general recursive formula finding N_d for all $d \geq 0$ in 1993. Deriving this recursion will be the subject of this talk and on the way we will build on the Gromov-Witten theory introduced in the seminar and give motivation for the introduction of quantum cohomology.

2 Quantum Cohomology

Let X be a smooth, projective, homogeneous variety. Observe that on $H^*(X)$ there is an associative cup product and a bilinear non-degenerate pairing $\langle -,-\rangle : H^*(X) \times H^*(X) \to H^*(X)$ given by

$$\langle \alpha, \beta \rangle := \int \alpha \cup \beta. \tag{2}$$

The cup product and the non-degenerate bilinear pairing $\langle -,-\rangle$ give $H^*(X)$ the structure of a Frobenius algebra, with unit the fundamental class $1_X \in H^0(X)$. We can generalize this structure by defining the multiplication:

$$\alpha_1 *_\beta \alpha_2 = \mathrm{ev}_{3*}(\mathrm{ev}_1^*(\alpha_1) \cup \mathrm{ev}_2^*(\alpha_2)), \tag{3}$$

where the moduli space considered is $\overline{\mathcal{M}}_{0,3}(X,\beta)$. We introduce a formal parameter q^β for each element $\beta \in H_2(X;\mathbb{Z})_+$ with rule $q^{\beta_1}q^{\beta_2} = q^{\beta_1+\beta_2}$. Thus we define

$$\alpha_1 * \alpha_2 = \sum_{\beta \in H_2(X)} (\alpha_1 *_\beta \alpha_2) q^\beta \tag{4}$$

and extend this product $\mathbb{Q}[[H_2(X;\mathbb{Z})_+]]$-linearly to $H^*(X) \otimes \mathbb{Q}[[H_2(X;\mathbb{Z})_+]]$. The coefficients are the genus-zero, three point Gromov-Witten invariants. That is, if $\{T_i\}_{i=0}^m$ is a basis for $H^*(X)$ then we can write the small quantum product as

$$\alpha_1 * \alpha_2 = \sum_{\beta \in H_2(X)} \langle \alpha_1, \alpha_2, T_i \rangle_{0,3,\beta} g^{ij} T_j q^\beta \text{ where } g_{ef} = \int_X T_e \cup T_f, \text{ and } g^{ef} := (g_{ef})^{-1}. \tag{5}$$

The resulting structure on this vector space $QH_s^*(X)$ is called the **small quantum cohomology** of X.

Theorem 2.1. *The small quantum cohomology $QH_s^*(X)$ is a Frobenius algebra with the same unit of $H^*(X)$.*

Example 2.2. $QH_s^*(\mathbb{P}^n) \cong \mathbb{Q}[H][[q]]/(H^{n+1} - q)$ where H is the class of a hyperplane in $H^2(\mathbb{P}^n)$.

The small quantum cohomology ring of $\mathbb{P}^n$ is a deformation of the usual cohomology ring that contains the 3-point information. Therefore the gromov witten numbers N_d do not appear in the small quantum cohomology of $\mathbb{P}^2$. If we want to go beyond the information of 3-point functions and consider n-point functions, we need to study an object known as the big quantum cohomology associated to X. We define the Gromov-Witten potential:

$$\Phi(\gamma) := \sum_{n \geq 3} \sum_\beta \frac{\langle \gamma^n \rangle_{0,n,\beta}}{n!} q^\beta \tag{6}$$

Here we use the notation that $\gamma^n = \gamma, \dots, \gamma$, n-times. Let $\gamma = \sum_i y_i T_i$, we obtain a formal power series in the y_i given by:

$$\Phi(y_0, \dots, y_m) = \sum_{\substack{n_0 + \cdots + n_m = n \\ \beta \in H_2(X;\mathbb{Z})}} \langle T_0^{n_0}, \dots, T_m^{n_m} \rangle_{0,n,\beta} \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}. \tag{7}$$

Remark 2.3. By [[FP97], Lem 15] we have that for any n, there are only finitely many β such that $\langle \gamma^n \rangle_{0,n,\beta}$ are nonzero.

Example 2.4. The Gromov-Witten potential for $\mathbb{P}^2$ with $\gamma = y_0 1 + y_1 H + y_2 H^2 \in H^*(\mathbb{P}^2)$ is given by

$$\Phi(\gamma) = \frac{1}{2}(y_0 y_1^2 + y_0^2 y_2) + \sum_{d=1}^\infty N_d \frac{y_2^{3d-1}}{(3d-1)!} e^{dy_1} \tag{8}$$

It will be convenient to define the following bit of notation $\Phi_{ijk} := \partial_i \partial_j \partial_k \Phi$ with $0 \leq i,j,k \leq m$.

Definition 2.5. Let X be a smooth projective variety, and let $\{T_i\}_i$ be a basis for $H^*(X)$. The **big quantum product** on $H^*(X)[[y_0, \dots, y_m, q]]$ is defined on the basis $T_0, \dots, T_m$ as

$$T_i *_b T_j := \sum_{e,f} \Phi_{ije} T^e, \text{ where } T^e = \sum_{f=1} g^{ef} T_f \tag{9}$$

where the product is then extended $\mathbb{Q}[[y_0, \ldots, y_m]]$-linearly to a product on the $\mathbb{Q}[[y_0, \ldots, y_m]]$-module $H^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}[[y_0, \ldots, y_m]]$ making it a $\mathbb{Q}[[y_0, \ldots, y_m]]$-algebra.

Remark 2.6. One should check that the product defined above is indeed well-defined. That is, given a basis $T'_0, \ldots, T'_m$, there is a linear change of coordinates from $H^*(X) \otimes \mathbb{Q}[[y_0, \ldots, y_m]]$ to $H^*(X) \otimes \mathbb{Q}[[y'_0, \ldots, y'_m]]$ identifying the two product structures.

Example 2.7. The big quantum product for $\mathbb{P}^2$ with basis $T_0 = 1, T_1 = [L]$ and $T_2 = [pt]$ for $H^*(\mathbb{P}^2)$ is given by

$$T^0 = T_2 = [pt] \quad T^1 = T_1 = [L] \text{ and } T^2 = T_0 = 1 = [\mathbb{P}^2]. \tag{10}$$

Using the potential for $\mathbb{P}^2$ defined above, we find

$$[L] * [L] = \sum_{k=0}^{2} \Phi_k T^k \ = \Phi_{110}[pt] + \Phi_{111}[L] + \Phi_{112}[\mathbb{P}^2] = \left(\int_{\mathbb{P}^2} [L] \cup [L] \right) [pt] + \Phi_{111}[L] + \Phi_{112}[\mathbb{P}^2]. \tag{11}$$

Using $\Phi_{110} = 1 = \int_{\mathbb{P}^2} [L] \cup [L]$ we see that we now have additional higher order terms to the classical cup product.

It's natural to ask what properties such a multiplication has, in particular whether it's associative or commutative. Well, commutativity is clear as the product is symmetric in the subscripts i.e $\phi_{ijk} = \Phi_{jik}$

$$T_j * T_i = \sum_k \Phi_{jik} T^k = \sum_k \Phi_{ijk} T^k = T_i * T_j. \tag{12}$$

For associativity, we have

$$(T_i * T_j) * T_k = \sum_{e,f} \Phi_{ije} g^{ef} T_f * T_k = \sum_{e,f} \sum_{c,d} \Phi_{ije} g^{ef} \Phi_{fkc} g^{cd} T_d \tag{13}$$

$$T_i * (T_j * T_k) = \sum_{e,f} \Phi_{jke} g^{ef} T_i * T_f = \sum_{e,f} \sum_{c,d} \Phi_{jke} g^{ef} \Phi_{ifc} g^{cd} T_d, \tag{14}$$

and since the matrix g^{cd} is non-singular the equality of $(T_i * T_j) * T_k = T_i * (T_j * T_k)$ is equivalent to the following non-linear partial differential equation

$$\sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl} = \sum_{e,f} \Phi_{jke} g^{ef} \Phi_{ifl}, \tag{15}$$

called the **Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation**. A (formal) solution to this nonlinear PDE is called a *Gromov-Witten potential.* One can also show $T_0 = 1$ is a unit for the $*$-multiplication.

Definition 2.8. The big quantum product endows $H^*(X) \otimes \mathbb{Q}[[y_0, \ldots, y_m]]$ with a multiplication. The ring

$$QH^*(X) := H^*(X) \otimes \mathbb{Q}[[y_0, \ldots, y_m]], \tag{16}$$

is called the **big quantum cohomology ring** of X.

Theorem 2.9 ([FP97], Thm 4)**.** *The biq quantum cohomology ring $QH^*(X)$ is a commutative, associate $\mathbb{Q}[[y_0, \ldots, y_m]]$-algebra with unit T_0. In other words it is a Frobenius algebra.*

Example 2.10. When $X = \mathbb{P}^2$ the big quantum cohomology ring of X has the identification

$$QH^*(\mathbb{P}^2) \cong \frac{\mathbb{Q}[[y_0, y_1, y_2]][Z]}{Z^3 - \Phi_{111} Z^2 - 2\Phi_{112} Z - \Phi_{122}}, \tag{17}$$

which we can compare with the cohomology ring $H^*_{\mathbb{Q}}(\mathbb{P}^2) \cong \mathbb{Q}[Z]/Z^3$ with basis $\{1, Z, Z^2\}$.

3 The Gromov-Witten Theory of $\mathbb{P}^2$

Recall the Gromov-Witten potential for $\mathbb{P}^2$, where $\gamma = y_0 1 + y_1 H + y_2 H^2 \in H^*(X)$ is given by

$$\Phi(\gamma) = \frac{1}{2}(y_0^2 y_2 + y_0 y_1^2) + \sum_{d=1}^{\infty} N_d \frac{e^{dy_1} y_2^{3d-1}}{(3d-1)!} q^d. \tag{18}$$

Using the associativity of the big quantum cohomology product, one can derive the WDVV equation for $\mathbb{P}^2$, which has the form

$$\Phi_{222} = \Phi_{112}^2 - \Phi_{111}\Phi_{122}. \tag{19}$$

Computing the term on the left hand side gives

$$\Phi_{222} = \sum_{d=1}^{\infty} N_d \frac{e^{dy_1} y_2^{3d-4}}{(3d-4)!} q^d. \tag{20}$$

On the right hand side we compute

$$\Phi_{112}^2 = \left(\sum_{d=1}^{\infty} d^2 N_d \frac{e^{dy_1} y_2^{3d-2}}{(3d-2)!} q^d \right)^2 \tag{21}$$

$$= \sum_{d=1}^{\infty} \sum_{d=d_1+d_2} d_1^2 d_2^2 N_{d_1} N_{d_2} \frac{e^{dy_1} y_2^{3d-4}}{(3d_1-2)!(3d_2-2)!} q^d, \tag{22}$$

and

$$\Phi_{111}\Phi_{122} = \partial_1^3 \Phi \cdot \partial_1 \partial_2^2 \Phi \tag{23}$$

$$= \left(\sum_{d=1}^{\infty} d^3 N_d \frac{e^{dy_1} y_2^{3d-1}}{(3d-1)!} q^d \right) \left(\sum_{d=1}^{\infty} d N_d \frac{e^{dy_1} y_2^{3d-3}}{(3d-3)!} q^d \right) \tag{24}$$

$$= \sum_{d=1}^{\infty} \sum_{d=d_1+d_2} d_1^3 d_2 N_{d_1} N_{d_2} \frac{e^{dy_1} y_2^{3d-3}}{(3d_1-1)!(3d_2-3)!}. \tag{25}$$

Equating coefficients of both sides for a fixed value of d we obtain

$$\frac{N_d}{(3d-4)!} = \sum_{d=d_1+d_2} N_{d_1} N_{d_2} \left[\frac{d_1^2 d_2^2}{(3d_1-1)!(3d_2-2)!} + \frac{d_1^3 d_2}{(3d_1-1)!(3d_2-3)!} \right]. \tag{26}$$

Rearranging and using $d_2 = d - d_1$, we derive Kontsevich's formula for the Gromov-Witten invariants of $\mathbb{P}^2$.

Theorem 3.1 (Kontsevich 1993)**.** *The number of rational curves N_d of degree d that pass through $(3d-1)$ points which are in general position in the $\mathbb{P}^2$, satisfies the recursion relation*

$$N_d = \sum_{d=d_1+d_2} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1 d_2 \binom{3d-4}{3d_1-1} \right]. \tag{27}$$

where $N_1 = 1$.

Computing N_d for $d = 1, \ldots, 8$ using Kontsevich's formula gives

$$N_1 = 1, \quad N_2 = 1, \quad N_3 = 12, \quad N_4 = 620, \quad N_5 = 87,304,$$

$$N_6 = 26,312,976, \quad N_7 = 14,616,808,192,$$

$$N_8 = 13,525,751,027,392.$$

References

[FP97] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology, 1997.