DAG Seminar: Derived Algebraic Stacks

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1. Introduction

1.1. **Motivation.** This lecture aims to introduce the main objects of study for the rest of the seminar, higher stacks following [Kha23]. Therefore we give many definitions in rapid succession with few examples left until the end.

1.2. **Derived Stacks.** Fix R a commutative ring. Recall from the end of last week that Fei described étale descent.

Definition 1.1. A derived stack is a functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ satisfying étale descent.

where $\mathrm{dCAlg}_R := \mathrm{Anim}(\mathrm{CAlg}_R)$ is the category of derived R-algebras. Let $\mathrm{ACRing} := \mathrm{Anim}(\mathrm{CRing})$, then in particular $\mathrm{dCAlg}_\mathbb{Z} \simeq \mathrm{ACRing}$. We denote the ∞-category of derived stacks by

$$\mathrm{DStk} := \mathrm{Shv}(\mathrm{dCAlg}_R^{\mathrm{op}}, \mathrm{Grpd}_\infty).$$

Example 1.2. An affine derived scheme over R is a derived stack, where $\mathrm{Spec}(A) : \mathrm{ACRing} \to \mathrm{Grpd}_\infty$ with $B \mapsto \mathrm{Maps}(A, B)$ corepresented by an animated ring $A \in \mathrm{dCAlg}_R$.

Definition 1.3. Let $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ be a derived stack, the restriction of X along $\mathrm{CRing} \hookrightarrow \mathrm{ACRing}$ is the functor $X_{cl} : \mathrm{CRing} \to \mathrm{Grpd}_\infty$ called the classical truncation of X.

For instance if X is a derived algebraic stack then in particular $X_{\mathrm{cl}} : \mathrm{CAlg}_R \to \mathrm{Grpd}$ is an algebraic stack. Moreover the classical truncation of the derived fiber product is the usual fiber product, that is

$$(X \times_Z^{\mathbf{R}} Y)_{\mathrm{cl}} \simeq X \times_Z Y.$$

Example 1.4. The classical truncation of a derived affine scheme over R is $\mathrm{Spec}(A)_{\mathrm{cl}} \simeq \mathrm{Spec}(\pi_0(A))$.

Remark 1.5. If the ∞-groupoid $X(A)$ is 1-truncated if for all $A \in \mathrm{dCAlg}_R$ then $X_{\mathrm{cl}} : \mathrm{CAlg} \to \mathrm{Grpd}$ is a stack.

1.3. **Derived Schemes.**

Definition 1.6. Let U and X be derived stacks, with morphism $j : U \to X$

(1) If U and X are affine j is an open immersion if it is étale ($\mathcal{O}_X \to \mathcal{O}_U$ is an étale morphism of derived R-algebras) and $U_{\mathrm{cl}} \to X_{\mathrm{cl}}$ is an open immersion (classically).
(2) If X is affine j is an open immersion if it is a monomorphism (the diagonal $U \to U \times_U U$ is an isomorphism) and there exists a collection of affines $(U_\alpha)_\alpha$ and a surjection $\sqcup_\alpha U_\alpha \to U$ such that $U_\alpha \to U \to X$ is an open immersion of affines.
(3) In general, the morphism j is an open immersion if for every affine S and every $S \to X$ the product $U \times_X S \to S$ is an open immersion to an affine.

Definition 1.7. A derived stack X is a derived scheme if there exists a collection $(U_\alpha \hookrightarrow X)_\alpha$ of open immersions where U_α are affine derived schemes, and a surjection $\coprod_\alpha U_\alpha \to X$.

Remark 1.8. A derived scheme X is 0-truncated, in the sense that the functor $X : \mathrm{dCAlg}_R \to \mathrm{Grpd}_\infty$ takes values in sets (= 0-truncated or discrete ∞-groupoids).

Definition 1.9. A morphism $f : X \to Y$ is schematic if for every affine V and every morphism $V \to Y$ the derived fibered product $X \times_V^{\mathbf{R}} Y$ is a derived scheme.

Definition 1.10. A schematic morphism $f : X \to Y$ of derived stacks is smooth (resp. étale) if for every affine V and every morphism $V \to Y$ there exists a collection of open immersions $(U_\alpha \to X \times_V Y)_\alpha$ where each U_α is affine and each composite

$$U_\alpha \to X \times_Y V \to V,$$

is a smooth (resp. étale) morphism of affines.

2. Derived Algebraic Stacks

We now define higher Artin stacks by induction:

Definition 2.1. A derived stack $X : \mathrm{ACRing} \to \mathrm{Grpd}_\infty$ is 0-Artin, or a derived algebraic space if

(1) the diagonal $X \to X \times X$ is schematic and a monomorphism;
(2) there exists an étale surjection $U \to X$ where U is a derived scheme.

Definition 2.2. A morphism $f : X \to Y$ is 0-Artin, or representable if for every affine V and every morphism $V \to Y$ the fibered product $X \times_Y^R V$ is a derived algebraic space (0-Artin).

Definition 2.3. A 0-Artin morphism $f : X \to Y$ is flat (resp. smooth, surjective) if for every affine V and every morphism $V \to Y$ there exists a derived scheme U and an étale surjection $U \to X \times_Y V$ such that the composition

$$U_\alpha \to X \times_Y V \to V,$$

is flat (resp. smooth, surjective).

For $n > 0$, inductively we define

Definition 2.4. For $n \geq 1$ a morphism of derived stacks $f : X \to Y$ is $(n-1)$-Artin if for every affine V and every morphism $V \to Y$ the fibered product $X \times_Y^R V$ is $(n-1)$-Artin.

Definition 2.5. A derived stack X is n-Artin if its diagonal is $(n-1)$-Artin and there exists a smooth surjection $U \to X$ where U is a derived scheme.

Definition 2.6. An $(n-1)$-Artin morphism is $f : X \to Y$ is flat (resp. smooth or surjective) if there exists a derived scheme U and a smooth surjection such that the composition

$$U \to X \times_Y V \to V$$

is flat (resp. smooth or surjective).

Following Gaitsgory we redefine Artin stacks to be higher Artin stacks, and algebraic stacks to be 1-Artin stacks.

Definition 2.7. A derived stack is Artin if it is n-Artin for some n.

Definition 2.8. A morphism $f : X \to Y$ of derived stacks is Artin if it is n-Artin for some n.

Definition 2.9. A morphism of derived stacks is flat (resp. smooth or surjective) if it is n-Artin and flat (resp. smooth or surjective) for some n.

Remark 2.10. An n-Artin stack takes values in n-groupoids i.e., in ∞-groupoids that are n-truncated.

Definition 2.11. A derived algebraic stack X over R is Deligne-Mumford if it admits an étale surjection $U \to X$ from a derived scheme U. Equivalently if its classical truncation $X_{\mathrm{cl}} : \mathrm{dCAlg}_R \to \mathrm{Grpd}$ is a Deligne-Mumford stack.

Artin Level	**Description**
0-Artin	Derived Algebraic Spaces
1-Artin + X_{cl} is DM	Derived Deligne-Mumford Stacks
1-Artin	Derived Algebraic Stacks

Mapping stacks give a large class of examples of derived algebraic stacks. Let X be a smooth and proper scheme over R.

Example 2.12. The moduli stack of perfect complexes over X is the derived stack $\mathcal{M}_{\mathrm{perf}(X)} = \underline{\mathrm{Maps}}(X, \mathcal{M}_{\mathrm{perf}})$. For $A \in \mathrm{dCAlg}_r$, its A-points are morphisms $X_A := X \times \mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{perf}}$ over $\mathrm{Spec}(A)$, i.e., perfect complexes on X_A.

Example 2.13. Let G be a smooth group scheme, the Moduli stack $\mathcal{M}_{\mathbf{Bun}_G(X)} = \underline{\mathrm{Maps}}(X, \mathrm{BG})$ of G-torsors (a.k.a principal G-bundles) over X is a derived algebraic stack. For $A \in \mathrm{dCAlg}_R$, it's A-points are morphisms $X_A \to \mathrm{BG}$ over $\mathrm{Spec}(A)$ i.e, G-torsors on X_A.

Example 2.14. The moduli stack of vector bundles on X is the substack $\mathcal{M}_{\mathbf{Vect}(X)} \subseteq \mathcal{M}_{\mathrm{perf}(X)}$ defined as follows: for $A \in \mathrm{dCAlg}_R$, an A-point of $\mathcal{M}_{\mathrm{perf}(X)}$ belongs to $\mathcal{M}_{\mathbf{Vect}(X)}$ if and only if the corresponding perfect complex $\mathcal{F} \in \mathrm{D}_{\mathrm{perf}}(X_A)$ is connective and flat over X_A.

References

[Kha23] Adeel A. Khan. Lectures on algebraic stacks, 2023.