Last time

Last time Adam talked about trying to understand moduli spaces of coherent sheaves on curves. The main characters we saw were the *charge* $Z : K_0(\mathbb{C}) \to \mathbb{C}$, defined as

$$Z(\mathcal{E}) = -\deg(\mathcal{E}) + i\operatorname{rank}(\mathcal{E}) \in \mathbb{C},$$

using the related quantities $\deg(\mathcal{E})$ the degree of $\mathcal{E}$, $\operatorname{rank}(\mathcal{E})$ the rank of $\mathcal{E}$, and $\mu(\mathcal{E}) = \deg(\mathcal{E})/\operatorname{rank}(\mathcal{E})$ the *slope* of $\mathcal{E}$. A key property over the curve C is that $\deg(\mathcal{E}) > 0$ whenever $\operatorname{rank}(\mathcal{E}) = 0$ (assuming $\mathcal{E} \neq 0$).

We called a non-zero coherent sheaf $\mathcal{E} \in \mathsf{Coh}(C)$: [Def 2.3]

1. *semistable* if for all subsheaves $\mathcal{F} \subsetneq \mathcal{E}$ it holds that $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$,
2. *stable* if for all subsheaves $\mathcal{F} \subsetneq \mathcal{E}$ it holds that $\mu(F) < \mu(\mathcal{E})$.

The main result underpinning everything is the existence (and uniqueness) of the Harder-Narasimhan filtration: [Thm 2.6]

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = E$$

of coherent sheaves such that each quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and $\mu(\mathcal{E}_1/\mathcal{E}_0) > \cdots > \mu(\mathcal{E}_n/\mathcal{E}_{n-1})$.

Today

Today, we shall see generalisations of the kind of stability encountered last week, to higher dimensions, and abelian categories.

First, briefly about some different types of stability that we won't really go into.

Gieseker-Maruyama-Simpson stability [Def 3.2]

There is the notion of Gieseker-Maruyama-Simpson stability for X a smooth projective variety of dimension $n \geqslant 2$.

If $\omega \in H^2(X, \mathbb{R})$ is an ample divisor class and $B \in H^2(X, \mathbb{R})$ is some other divisor class, then one can define the Chern character twisted by B as

$$\operatorname{ch}^B(\mathcal{E}) = \operatorname{ch}(\mathcal{E}) \cdot e^{-B} = \operatorname{ch}(\mathcal{E})\left(1 - B + \frac{1}{2}B^2 - \frac{1}{6}B^3 + \cdots\right).$$

The B-twisted Hilbert polynomial of pure sheaf $\mathcal{E} \in \mathsf{Coh}(X)$ of dimension d is then

$$P(\mathcal{E}, B, t) = \int_X \operatorname{ch}^B(\mathcal{E}) \cdot e^{t\omega} \cdot \operatorname{td}_X = a_d(\mathcal{E}, B)t^d + \cdots + a_0(\mathcal{E}, B),$$

where td_X is the Todd class of X.

One then says that $\mathcal{E}$ is B-twisted:

1. *semistable* if for all subsheaves $\mathcal{F} \subsetneq \mathcal{E}$ it holds that $P(\mathcal{F}, B, t)/a_d(\mathcal{F}, B) \leqslant P(\mathcal{E}, B, t)/a_d(\mathcal{E}, B)$ for $t \gg 0$,
2. *stable* if for all subsheaves $\mathcal{F} \subsetneq \mathcal{E}$ it holds that $P(\mathcal{F}, B, t)/a_d(\mathcal{F}, B) < P(\mathcal{E}, B, t)/a_d(\mathcal{E}, B)$ for $t \gg 0$.

One has analogous results to those presented last week in this case, and we get a moduli space parametrising semistable vector bundles.

Twisted slope stability [Def 3.3]

There is also a B-twisted version of slope stability, defined for $\mathcal{E} \in \mathsf{Coh}(X)$ by

$$\begin{aligned}\mu_{\omega,B}(\mathcal{E}) = \frac{\int_X \omega^{n-1} \cdot \mathrm{ch}_1^B(\mathcal{E})}{\int_X \omega^n \cdot \mathrm{ch}_0^B(\mathcal{E})} &= \frac{\int_X \omega^{n-1} \cdot \mathrm{ch}_1(\mathcal{E})}{\int_X \omega^n \cdot \mathrm{ch}_0(\mathcal{E})} - \frac{\int_X \omega^{n-1} \cdot B}{\int_X \omega^n} \\ &= \underbrace{\frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}}_{\mu(\mathcal{E})} - \frac{\int_X \omega^{n-1} \cdot B}{\int_X \omega^n}.\end{aligned}$$

One says that a coherent sheaf $\mathcal{E} \in \mathsf{Coh}(C)$ is

1. *semistable* if for all subsheaves $\mathcal{F} \subsetneq \mathcal{E}$ it holds that $\mu_{\omega,B}(\mathcal{F}) \leqslant \mu_{\omega,B}(\mathcal{E})$,
2. *stable* if for all subsheaves $\mathcal{F} \subsetneq \mathcal{E}$ it holds that $\mu_{\omega,B}(F) < \mu_{\omega,B}(\mathcal{E})$.

We can see that the definition of (semi)stability doesn't depend on B, and it's the one we had previously when we set $B = 0$. In this context, one also gets a Harder-Narasimhan filtration.

Bridgeland stability

Bridgeland stability is the type of stability we will be focusing on. In order to generalise slope stability, we will change the category we are working in from coherent sheaves on X to other abelian categories that live inside $\mathcal{D}^b(X)$, the bounded derived category of coherent sheaves on X.

Stability in abelian categories

From now, we will let $\mathcal{A}$ denote an abelian category, and $K_0(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$. We begin with a generalisation of the *charge* and friends.

Definition (Stability function) [Def 4.1]

Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be an additive homomorphism. Write

$$Z(E) = -D(E) + i\,R(E),$$

where R and D are called the *generalised rank* and *degree* respectively, and $\mathrm{M} = D/R$ the *generalised slope*. We say that Z is a stability function if it satisfies an analogous property to the rank and degree from before:

1. $R(E) \geqslant 0$, and
2. if $D(E) = 0$ then $R(E) > 0$.

Definition (Stability) [Def 4.2]

We shall say a non-zero object $E \in \mathcal{A}$ is

1. *semistable* if for all proper non-trivial subobjects $F \subsetneq E$ it holds that $\mathrm{M}(F) < \mathrm{M}(E)$,
2. *stable* if for all proper non-trivial subobjects $F \subsetneq E$ it holds that $\mathrm{M}(F) \leqslant \mathrm{M}(E)$.

Equivalently, one has that $E \in \mathcal{A}$ is

1. *semistable* if for all quotients $E \twoheadrightarrow G$ it holds that $\mathrm{M}(E) < \mathrm{M}(G)$,
2. *stable* if for all quotients $E \twoheadrightarrow G$ it holds that $\mathrm{M}(E) \leqslant \mathrm{M}(G)$.

Funny exercise [Lem 4.5]

If $A, B \in \mathcal{A}$ are non-zero objects which are semistable with $\mathrm{M}(A) > \mathrm{M}(B)$, then $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$.

Proof

Let $f : A \to B$ be a non-trivial morphism, and $0 \neq Q = \mathrm{im}\, f \subset B$. But now $\mathrm{M}(A) \leqslant \mathrm{M}(Q) \leqslant \mathrm{M}(B)$, contradicting $\mathrm{M}(A) > \mathrm{M}(B)$.

Definition (Stability condition) [Def 4.6]

Of course, we don't get the Harder-Narasimhan filtration for free, so we have to ask for it. One calls $(\mathcal{A}, Z)$ a *stability condition* if

1. Z is a stability function, and
2. Any non-zero $E \in \mathcal{A}$ has a Harder-Narasimhan filtration:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

   such that each E_i/E_{i-1} is semistable and $\mathrm{M}(E_1/E_0) > \cdots > \mathrm{M}(E_n/E_{n-1})$.

In such a case, the filtration is always unique up to isomorphism (what isomorphism?). Typically, one will need to prove that Harder-Narasimhan filtrations exist. The following result provides a criterion for their existence.

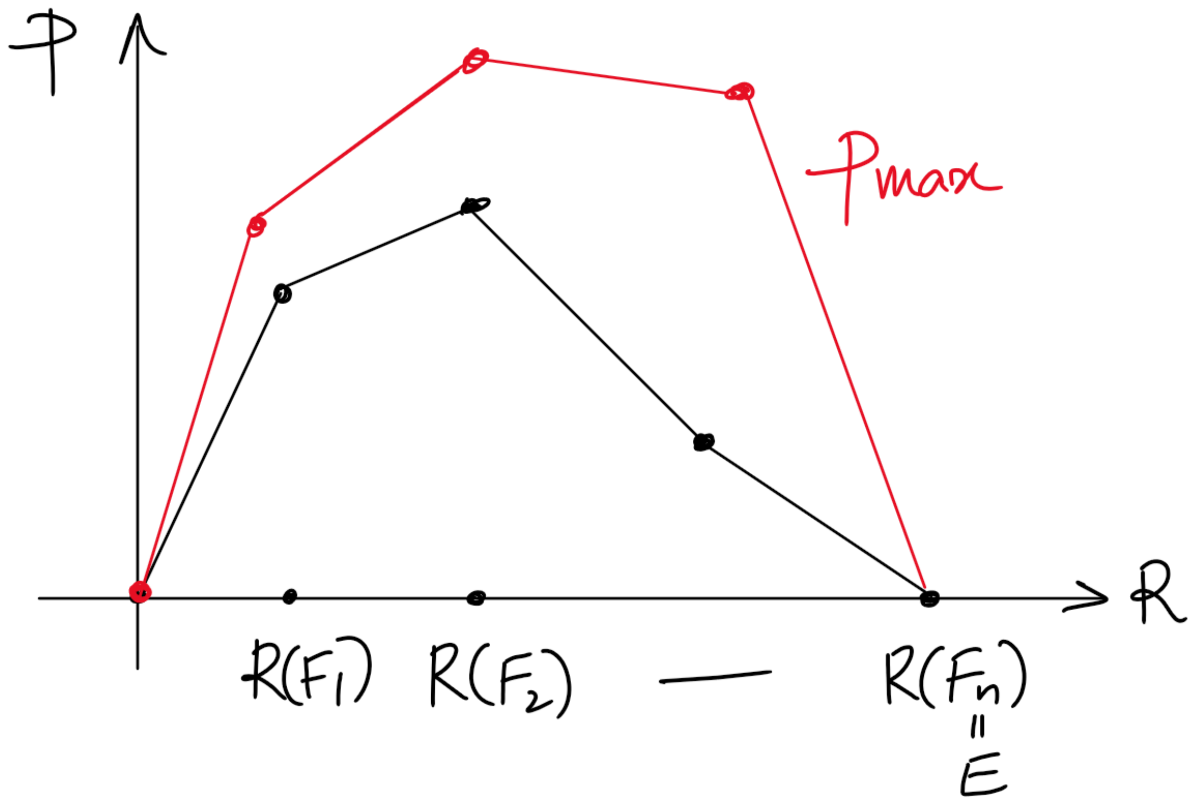
The result

If $\mathcal{A}$ is Noetherian and the image of R (the generalised rank) is discrete in $\mathbb{R}$, then:

1. **[Lem 4.9]** For any object $E \in \mathcal{A}$, there exists a number $D_E \in \mathbb{R}$ such that for any

subobject $F \subset E$ it holds that $D(F) \leqslant D_E$.
2. **[Prop 4.10]** (Using Lem 4.9) Harder-Narasimhan filtrations exist in $\mathcal{A}$.

Draw picture:



Here, we are looking at a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = E$$

of E, and the function p being plotted above is defined

$$p(F) = D(F) - R(F)\mathrm{M}(E).$$

Lem 4.9 gives the existence of a $p_{\max}$, and one proves that it is a Harder-Narasimhan filtration.

Definition (Noetherian abelian category) [Def 4.8]

We say that $\mathcal{A}$ is *Noetherian* if for any $A \in \mathcal{A}$ and ascending chain of subobjects

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A,$$

we have $A_i = A_j$ for all $i, j \gg 0$.

Examples

1. Curves

When C is a curve and $\mathcal{A} = \mathsf{Coh}(C)$, the charge function

$$Z(\mathcal{E}) = -\deg(\mathcal{E}) + i\,\mathrm{rank}(\mathcal{E})$$

is a stability function, and (semi)stability is the same as last week.

2. Higher dimension

When X is a smooth projective variety of dimension $n \geqslant 2$, recall the setup we had before with $\omega, B \in H^2(X, \mathbb{R})$ the ample and possibly not divisor classes. Letting $\mathcal{A} = \mathsf{Coh}(X)$, we get

$$\overline{Z}_{\omega,B}(\mathcal{E}) = -\int_X \omega^{n-1} \cdot \mathrm{ch}_1^B(\mathcal{E}) + i \int_X \omega^n \cdot \mathrm{ch}_0^B(\mathcal{E})$$

from our example before is **NOT** a stability function. It does not satisfy the condition:

- $R(\mathcal{E}) = 0$ implies $D(\mathcal{E}) > 0$: a torsion sheaf supported in codimension $\geqslant 2$ has 0 degree.

3. Alternative

Instead, take the quotient category

$$\mathcal{A} = \mathsf{Coh}_{n,n-1}(X) = \mathsf{Coh}(X)/\,\mathsf{Coh}(X)_{\leqslant n-2}$$

where $\mathsf{Coh}(X)_{\leqslant n-2}$ is the subcategory of coherent sheaves supported in dimension $\leqslant n-2$, then $\overline{Z}_{\omega,B}$ above does give a stability function, giving the same sense of stability as the twisted slope stability we saw at the beginning.